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Research Article

Alternative Proof of the Ribbonness on Classical Link

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Abstract

Alternative proof is given for an earlier presented result that if a link in 3-space bounds a compact oriented proper surface (without closed component) in the upper half 4-space, then the link bounds a ribbon surface in the upper half 4-space which is a boundary-relative renewal embedding of the original surface.

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More Information

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1. Introduction

For a set A in the 3-space $\mathbf{R}^3 = \{(x, y, z) | -\infty < x, y, z < +\infty\}$ and an interval

$J \subset \mathbf{R}$, let

$AJ = \{(x, y, z, t) | (x, y, z) \in A, t \in J\}$.

The *upper-half 4-space* $\mathbf{R}^4_+$ is denoted by $\mathbf{R}^3[0,+\infty)$. Let k be a link in the 3- space $\mathbf{R}^3$, which always bounds a compact oriented proper surface F embedded smoothly in the upper-half 4-space $\mathbf{R}^4_+$, where $\mathbf{R}^3[0]$ is canonically identified with $\mathbf{R}^3$. Two such surfaces F and F' in $\mathbf{R}^4_+$ are *equivalent* if there is an orientation-preserving diffeomorphism f of $\mathbf{R}^4_+$ sending F to F', where f is called an *equivalence*. For a link k_0 in $\mathbf{R}^3$, let $\mathbf{b}$ be a band system spanning k_0, namely a system of finitely many disjoint oriented bands spanning the link k_0 in $\mathbf{R}^3$. The pair $(k_0, \mathbf{b})$ is called a *banded link*. The *surgery link* of $(k_0, \mathbf{b})$ is the link obtained from k_0 by surgery along $\mathbf{b}$. Assume that the surgery link of a banded link $(k_0, \mathbf{b})$ is a trivial link κ in $\mathbf{R}^3$. Then the band system $\mathbf{b}$ is considered as a band system β spanning κ. The pair (κ, β) is called a *banded loop system* with *loop system* κ and *surgery link* k_0. Throughout the paper, the surgery link k_0 will be a union $k \cup \mathbf{o}$ of a link k in question and a trivial link $\mathbf{o}$ called an *extra trivial link*. Here, it is assumed that there is a band sub-system $\mathbf{b}_1$ of the band system $\mathbf{b}$ such that $\mathbf{b}_1$ connects to $\mathbf{o}$ with just one band $b_1 \in \mathbf{b}_1$ for every component $o \in \mathbf{o}$ and every band $b \in \mathbf{b}_1^C = \mathbf{b} \setminus \mathbf{b}_1$ spans the link k. Let $\boldsymbol{\alpha}_1$ be the arc system of the attaching arc α_1 of every band $b_1 \in \mathbf{b}_1$ to $o \in \mathbf{o}$, and α_1^C the complementary arc system of $\boldsymbol{\alpha}_1$ in $\mathbf{o}$ consisting of every complementary arc $\boldsymbol{\alpha}_1^C = cl(0 \setminus \boldsymbol{\alpha}_1)$. Any disk system $\mathbf{d}$ in $\mathbf{R}^3$ bounded by the extra trivial link $\mathbf{o}$ is called an *extra disk system*, which is fixed and the argument proceeds. Let $\boldsymbol{\delta}$ be a disk system consisting of disjoint disks in $\mathbf{R}^3$ with $\partial\boldsymbol{\delta} = \boldsymbol{\kappa}$, which is called a *based disk system* for a loop system $\boldsymbol{\kappa}$. A ribbon surface-link cl(F^1_{-1}) in $\mathbf{R}^4$ is constructed from a banded loop system $(\boldsymbol{\kappa}, \boldsymbol{\beta})$ by taking the surgery of the trivial S^2- link

$$O = \partial(\boldsymbol{\delta}[-1, 1]) = \boldsymbol{\delta}[-1] \cup (\partial\boldsymbol{\delta})[-1, 1] \cup \boldsymbol{\delta}[1]$$

along the 1-handle system $\beta[-t, t]$ in $\mathbf{R}^4$ for any t with $0 < t < 1$. The proper surface ucl$\left(F^1_0\right) = cl(F^1_{-1}) \cap R^4_+$ in $\mathbf{R}^4_+$ is called the *upper-closed realizing surface* of a banded loop system $(\boldsymbol{\kappa}, \boldsymbol{\beta})$ with surgery link k_0. Note that choices of the based disk systems $\boldsymbol{\delta}$ are independent of the equivalences of ucl(F^1_0) and cl(F^1_{-1}) by Horibe-Yanagawa's lemma [1]. The reason for dealing with a banded loop system $(\boldsymbol{\kappa}, \boldsymbol{\beta})$ rather than a banded link $(k_0, \mathbf{b})$ is because not only can a based disk system $\boldsymbol{\delta}$ be chosen freely, but it also makes a band deformation of the band system $\boldsymbol{\beta}$ easier. Actually, an isotopic deformation of $\boldsymbol{\beta}$ respecting the arc system $\boldsymbol{\alpha}_1$ and the loop system $\boldsymbol{\kappa}$ does not change the ribbon surface-link cl(F_{-1}^{1}) in $\mathbf{R}^4$ and the proper surface ucl(F_0^1) in $\mathbf{R}_+^4$, up to equivalences.

Let cl(F_{-1}^{1})$_{\mathbf{d}}$ be the surface-link in $\mathbf{R}^4$ obtained from the ribbon surface-link cl(F_{-1}^{1}) by surgery along the 2-handle $\mathbf{d}[-\varepsilon, \varepsilon]$ on cl(F_{-1}^{1}) where $0 < \varepsilon < t < 1$.

The proper surface $P\left(F_0^1\right) = cl(F_{-1}^1)d \cap \mathbf{R}_+^4$ in $\mathbf{R}_+^4$ with $\partial P = F_0^1 = k$ is called a *proper realizing surface* of a banded loop system (κ, β) with surgery link $k_0 = k \cup \mathbf{o}$. The following theorem is known [1].

Normal form theorem: Every compact oriented proper surface F without closed component in the upper-half 4-space $\mathbf{R}_+^4$ with $\partial F = k$ in $\mathbf{R}^3$ is equivalent to a proper realizing surface $P\left(F_0^1\right)$ in $\mathbf{R}_+^4$ with $\partial P = \left(F_0^1\right) = k$ of a banded loop system (κ, β) with surgery link $k_0 = k + \mathbf{o}$ which is a split sum of k and an extra trivial link $\mathbf{o}$.

The proper realizing surface $P\left(F_0^1\right)$ in $\mathbf{R}_+^4$ is called a *normal form* of the proper surface F in $\mathbf{R}_+^4$. If the extra trivial link $\mathbf{o}$ is taken the empty link, namely $P\left(F_0^1\right) = ucl(F_0^1)$, then the proper surface F in $\mathbf{R}_+^4$ is called a *ribbon surface*. In the following example, it is observed that there are lots of compact oriented proper surfaces without closed component in $\mathbf{R}_+^4$ which are not equivalent to any ribbon surface in $\mathbf{R}_+^4$.

Example. For every link k in $\mathbf{R}^3$, let F' be any ribbon surface in $\mathbf{R}_+^4$ with $k = \partial F'$. For example, let F' be a proper surface in $\mathbf{R}_+^4$ obtained from a Seifert surface for k in $\mathbf{R}^3$ by an interior push into $\mathbf{R}_+^4$. Take a connected sum $F = F' \# K$ of F' and a nontrivial S^2-knot K in $\mathbf{R}^4$ with non-abelian fundamental group. Then $k = \partial F' = \partial F$. It is shown that F is not equivalent to any ribbon surface in $\mathbf{R}_+^4$. The fundamental groups of k, F', F, K are denoted as follows.

$\pi(k) = \pi_1(\mathbf{R}^3 \setminus k, x_0)$, $\pi(F') = \pi_1(\mathbf{R}^4 \setminus F', x_0)$, $\pi(F) = \pi_1(\mathbf{R}^4 \setminus F, x_0)$, $\pi(K) = \pi_1(S^4 \setminus K, x_0)$.

Let $\pi(k)^*$, $\pi(F')^*$, $\pi(F')^*$, $\pi(K)^*$ be the kernels of the canonical epimorphisms from the groups $\pi(k)$, $\pi(F')$, $\pi(F)$, $\pi(K)$ to the infinite cyclic group sending every meridian element to the generator, respectively. It is a special feature of a ribbon surface F' that the canonical homomorphism $\pi(k) \to \pi(F')$ is an epimorphism, so that the induced homomorphism $\pi(k)^* \to \pi(F')^*$ is onto. On the other hand, the canonical homomorphism $\pi(k) \to \pi(F)$ is not onto, because the group $\pi(F)^*$ is the free product $\pi(F')^* * \pi(K)^*$ and $\pi(K)^* \neq 0$ and the image of the induced homomorphism $\pi(k)^* \to \pi(F)^*$ is just the free product summand $\pi(F')^*$. Thus, the proper surface F in $\mathbf{R}_+^4$ is not equivalent to any ribbon surface.

A compact oriented proper surface F' in $\mathbf{R}_+^4$ is a *renewal embedding* of a compact oriented proper surface F in $\mathbf{R}_+^4$ if there is an orientation-preserving surfacediffeomorphism $F' \to F$ keeping the boundary fixed. A renewal embedding F' of F is *boundary-relative* if the link $k' = \partial F'$ in $\mathbf{R}^3$ is equivalent to the link $k = \partial F$ in $\mathbf{R}^3$. The proof of the following theorem is given [2]. In this paper, an alternative proof of this theorem is given from a viewpoint of deformations of a ribbon surface-link in $\mathbf{R}^4$.

Classical ribbon theorem: Assume that a link k in the 3-space $\mathbf{R}^3$ bounds a compact oriented proper surface F without closed component in the upper-half 4space $\mathbf{R}_+^4$. Then the link k in $\mathbf{R}^3$ bounds a ribbon surface F' in $\mathbf{R}_+^4$ which is a boundary-relative renewal embedding of F.

A link k in $\mathbf{R}^3$ is a *slice link in the strong sense* if k bounds a proper disk system embedded smoothly in $\mathbf{R}_+^4$. A link k in $\mathbf{R}^3$ is a *ribbon link* if k bounds a ribbon disk system in $\mathbf{R}_+^4$. The following corollary is a special case of Classical ribbon theorem.

Corollary 1: Every slice link in the strong sense in $\mathbf{R}^3$ is a ribbon link.

Thus, Classical ribbon theorem solves *Slice-Ribbon Problem*, [3,4]. The following corollary is obtained from Corollary 1.

Corollary 2: A link k in $\mathbf{R}^3$ is a ribbon link if a ribbon link is obtained from the split sum $k + \mathbf{o}$ of k and a trivial link $\mathbf{o}$ by a band sum of k and every component of $\mathbf{o}$.

The proof of the classical ribbon theorem is done throughout the section 2. An idea of the proof is to consider the 2-handle pair system $(D \times I, D' \times I)$ on the ribbon surface-link cl(F_{-1}^{1}) with $k + \mathbf{o}$ as the middle-cross sectional link such that $P\left(F_0^1\right)$ is equivalent to a previously given surface F in $\mathbf{R}_+^4$, where the 2-handle system $D \times I$ is constructed from the band system $\mathbf{b}_1$ and the 2-handle system $D' \times I$ is constructed from the extra disk system $\mathbf{d}$. The interior intersections of $(D \times I, D' \times I)$ will be eliminated and $(D \times I, D' \times I)$ becomes an O2-handle pair system on a new ribbon surface-link cl(F_{-1}^{1}) with $k + \mathbf{o}$ as the middle-cross sectional link obtained by sacrificing equivalences. Then $P\left(F_0^1\right)$ is a ribbon surface that is a boundary-relative renewal embedding of F, which will complete the proof.

2. Proof of classical ribbon theorem

Throughout this section, the proof of the classical ribbon theorem is done. Let F be a compact oriented proper surface without closed component in $\mathbf{R}^4_+$, and $\partial F = k$ a link in $\mathbf{R}^3$. By the normal form theorem, there is a banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with surgery link $k_0 = k + \mathbf{o}$ such that $P\left(F_0^1\right)$ is equivalent to F. The extra trivial link $\mathbf{o}$ is uniquely specified by the banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$), which is the union of the arc system $\boldsymbol{\alpha}_1$ and the complementary arc system α_1^c, where the interior of $\boldsymbol{\alpha}_1$ transversely meets the interior of a based disk system δ with finite points and is disjoint from the based loop system $\boldsymbol{\kappa}$ and α_1^c belongs to the loop system κ.

A *renewal embedding* of a banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with surgery link $k_0 = k \cup \mathbf{o}$ is a banded loop system ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) with surgery link $k_0' = k' \cup o$ such that there is a homeomorphism $\boldsymbol{\kappa} \cup \boldsymbol{\beta} \rightarrow \boldsymbol{\kappa}' \cup \boldsymbol{\beta}'$ with restrictios $\boldsymbol{\kappa} \rightarrow \boldsymbol{\kappa}'$ and $\boldsymbol{\beta} \rightarrow \boldsymbol{\beta}'$ orientationpreserved.

The following observation is directly obtained by definition.

(2.1) If a banded loop system system ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) with surgery link $k' \cup \mathbf{o}$ is a renewal embedding of a banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with surgery link $k \cup \mathbf{o}$, then the upper-closed realizing surface ucl$\left(F_0^1\right)'$ constructed from ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) is a renewal embedding of the upper-closed realizing surface ucl$\left(F_0^1\right)$ constructed from ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) such that $\partial ucl = \left(F_0^1\right) = k \cup o$ and $\partial ucl = \left(F_0^1\right)' = k' \cup o$.

A *transversal arc* of a band spanning a link is a simple proper arc in the band which is parallel to an attaching arc. For a band $b \in \mathbf{b}$ transversely meeting the interior of an extra disk $d \in \mathbf{d}$, the *d-arc system* of b is the arc system $d(b)$ of every transversal arc a of b in the interior of d. The $\mathbf{d}$-arc system of a band system $\mathbf{b}$ is the collection $\mathbf{d}(\mathbf{b})$ of $d(b)$ for every $d \in \mathbf{d}$ and every $b \in \mathbf{b}$. For a based disk $\delta \in \delta$, the *δ-arc system* of a band $\beta \in \beta$ is the arc system $\delta(\beta)$ of every transversal arc c of β in the interior of δ. The $\boldsymbol{\delta}$-arc system of β is the collection $\delta(\beta)$ of $\delta(\beta)$ for every $\delta \in \boldsymbol{\delta}$ and every $\beta \in \boldsymbol{\beta}$. A *normal proper arc* in the extra disk system $\mathbf{d}$ is a simple proper arc in $\mathbf{d}$ with the endpoints in the interior of the arc system $\boldsymbol{\alpha}_1$. The following assertion is shown.

(2.2) By isotopic deformations in $\mathbf{R}^3$, the banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) in $\mathbf{R}^3$ with surgery link $k_0 = k + \mathbf{o}$ is deformed so that a based disk system $\boldsymbol{\delta}$ transversely meets the extra disk system $\mathbf{d}$ with interior simple arcs or normal proper arcs in $\mathbf{d}$ except for the complementary arc system α_1^c.

Proof of (2.2). By transverse regularity, the intersection $d \cap \delta$ for every $d \in \mathbf{d}$ and every $\delta \in \boldsymbol{\delta}$ is made interior simple loops, interior simple arcs, clasp type simple arcs or simple proper arcs in $\mathbf{d}$ except for the complementary arc system α_1^c. A simple loop is changed into a normal proper arc by a pushing out deformation to $\boldsymbol{\alpha}_1$, Figure 1(1). A clasp type simple arc is changed into a simple proper arc by moving out the interior point to α_1, Figure 1(2). A simple proper arc

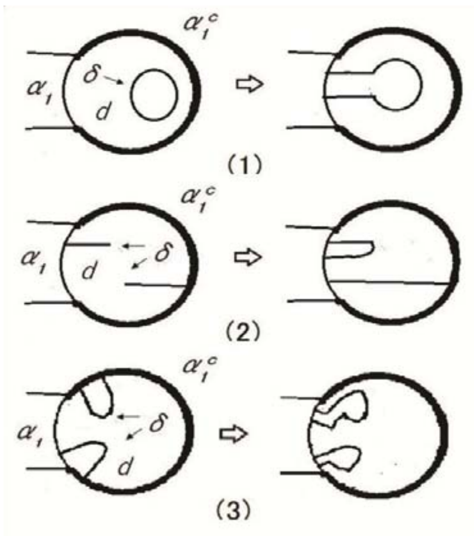

Figure 1: Changing the intersection of a based disk and an extra disk.

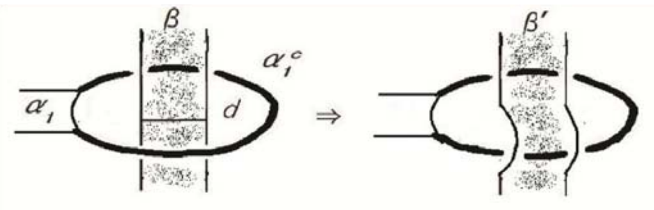

Figure 2: Band Move Operation.

which is not normal is also changed into a normal proper arc by a pushing out deformation of the arc system of $\boldsymbol{\delta}$ meeting a boundary collar of α_1^c in $\mathbf{d}$, Figure 1(3). Thus, a deformed based disk system $\boldsymbol{\delta}$ transversely meets $\mathbf{d}$ with interior simple arcs or normal proper arcs in $\mathbf{d}$ except for the complementary arc system α_1^c. This completes the proof of (2.2).

In the proof of (2.2), there is no need to worry about the intersection of the based disk system δ and the interior of the arc system $\boldsymbol{\alpha}_1$ in $\mathbf{R}^3$, because δ is taken and deformed in the 3-spaces $\mathbf{R}^3[\pm 1]$ and the extra disk system $\mathbf{d}$ and the arc system $\boldsymbol{\alpha}_1$ are taken and fixed in the 3-space $\mathbf{R}^3[0]$ for the ribbon surface-link cl(F_{-1}^1) in $\mathbf{R}^4$. The following operation gives a standard renewal embedding of a banded loop system.

Band move operation: In the banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with surgery link $k_0 = k \cup \mathbf{o}$, assume that there is a transversal arc c of a band $\beta \in \boldsymbol{\beta}$ in the interior of an extra disk $d \in \mathbf{d}$ and there is a simple path ω in d from a point $p \in c$ to an interior point of the arc $\boldsymbol{\alpha}_1^c = \partial d \cap \boldsymbol{\alpha}_1^c$ which avoids meeting β other than c. Let β' be a band obtained from β by sliding the arc c off the disk d along the path ω. Replace the banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with the banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}'$) obtained by replacing β with β', Figure 2.

By this operation, the new banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}'$) is a renewal embedding of the original banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$)

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and has as the surgery link a new union k_0' of the same links k and **o**, not necessarily the split sum k + **o**, because the band system $\boldsymbol{\beta}'$ is isotopic to $\boldsymbol{\beta}$ if α_1^c is forgotten. In the final stage of this paper, the surgery link k_0' will have $k \cap \mathbf{d} = \emptyset$, so that k_0' will be the split sum k + **o**, because **o** = ∂**d**.

To achieve a situation where the Band Move Operation can be applied, the follwing concept is needed. A *splitting* of a banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) is a banded loop system ($\boldsymbol{k}^*$, $\boldsymbol{\beta}^*$) such that a based disk system $\boldsymbol{\delta}^*$ for $\boldsymbol{\kappa}^*$ is obtained from a based disk system $\boldsymbol{\delta}$ for $\boldsymbol{\kappa}$ by splitting along a disjoint proper arc system γ in δ not meeting **o** and β, and the band system β^* is obtained from the band system β by adding the band system β_γ thickening γ. This splitting operation comes from Fission-Fusion move of a banded loop system, [5,6]. After some splittings of a banded loop system, a situation where the Band Move Operation can be applied is realized by a replacement of the based disk system and an isotopic deformation of the band system.

The following assertion is used.

(2.3) If there is a splitting ($\boldsymbol{k}^*$, $\boldsymbol{\beta}^*$) of a banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with surgery knot k_0 a union of k and **o** such that κ^* does not meet the interior of the extra disk system **d**, then there is a renewal embedding ($\boldsymbol{k}'$, $\boldsymbol{\beta}'$) of ($\boldsymbol{k}$, $\boldsymbol{\beta}$) such that ($\boldsymbol{k}'$, $\boldsymbol{\beta}'$) does not meet the interior of **d** and has the surgery knot $k_0' = k + \mathbf{o}$.

Proof of (2.3). Since κ^* does not meet the interior of **d**, there is a based disk system $\boldsymbol{\delta}^*$ for $\boldsymbol{\kappa}^*$ not meeting the interior of **d**. The band system $\boldsymbol{\beta}^*$ transversely meets the interior of **d** with transverse arc system A. Let δ_1^* be the sub-system of $\boldsymbol{\delta}^*$ containing the complementary arc system α_1^c in the boundary, and $N(\alpha_1^c)$ a boundary collar disk system of α_1^c in δ_1^*. The Band Move Operation means that the band system $\boldsymbol{\beta}^*$ is deformed so that the transverse arc system A moves from the interior of **d** into the interior of $N(\alpha_1^c)$. Then by changing the band system β_γ back into the arc system γ, the banded loop system ($\boldsymbol{k}^*$, $\boldsymbol{\beta}^*$) is changed back to a pair ($\boldsymbol{k}'$, $\boldsymbol{\beta}'$), where the loop system κ' bounds an immersed disk system δ' obtained from the based disk system $\boldsymbol{\delta}$ by moving a transverse arc system of β_γ into the interior of $N(\alpha_1^c)$. The immersed disk system δ' is deformed into a disjoint disk system by repeatedly pulling the band in β_γ connecting to an outer most disk of δ^* or passing the outer most disk of δ^* through $N(\alpha_1^c)$ in order to eliminate the nearest transverse arc of the band. This means that the loop system κ' is a trivial link and ($\boldsymbol{k}'$, $\boldsymbol{\beta}'$) is a banded loop system. Thus, there is a renewal embedding ($\boldsymbol{k}'$, $\boldsymbol{\beta}'$) of ($\boldsymbol{k}$, $\boldsymbol{\beta}$) which does not meet the interior of **d**. The surgery knot k_0' is necessarily the split sum k + **o** since ∂**d** = **o**. This completes the proof of (2.3).

By using (2.2) and (2.3), the following assertion is shown.

(2.4) There is a renewal embedding ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) of every banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) in $\mathbf{R}^3$ with surgery link k_0 = k + **o** such that ($\boldsymbol{k}'$, $\boldsymbol{\beta}'$) does not meet the interior of **d** and has the surgery knot $k_0' = k + \mathbf{o}$.

Proof of (2.4). By (2.2), a based disk system $\boldsymbol{\delta}$ of $\boldsymbol{\kappa}$ transversely meets the extra disk system **d** with interior simple arcs or normal proper arcs in **d** except for the complementary arc system α_1^c. Let A be the interior arc system which is made disjoint from $\boldsymbol{\beta}$ by isotopic deformations of $\boldsymbol{\beta}$ respecting the arc system $\boldsymbol{\alpha}_1$ and the loop system $\boldsymbol{\kappa}$. By taking a splitting of ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) along A, it is considered that the based disk system $\boldsymbol{\delta}$ transversely meets **d** only with normal proper arcs in **d** except for α_1^c. Then $\boldsymbol{\kappa}$ does not meet the interior of the extra disk system **d**. By (2.3), the proof of (2.4) is completed.

Let ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) be a banded loop system a banded loop system with surgery link k_0 = k + **o** such that $P\left(F_0^1\right)$ is equivalent to F. By (2.4), there is a renewal embedding ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) such that ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) does not meet the interior of the extra disk system **d**, and has the surgery link k + **o**. Let $\mathbf{b}'$ be the band system dual to the band system $\boldsymbol{\beta}'$, and $\mathbf{b}_1'$ the band sub-system of $\mathbf{b}'$ such that $\mathbf{b}_1'$ connects to **o** with just one band for every component of **o**. Let $\mathbf{b}_2' = \mathbf{b}' \setminus \mathbf{b}_1'$. Since $\mathbf{b}_1'$ does not meet the interior of **d**, the surgery link of the banded link $(k+o, \mathbf{b}')$ is equivalent to the link k and the upper-closed realizing surface ucl$\left(F_0^1\right)'$ of the banded link $(k, \mathbf{b}_2')$ is equivalent to the proper realizing surface $P\left(F_0^1\right)'$ of ($\boldsymbol{\kappa}'$, $\boldsymbol{\beta}'$) which is a ribbon surface in $\mathbf{R}_+^4$ and is a renewal embedding of the proper realizing surface $P\left(F_0^1\right)$ of the banded loop system ($\boldsymbol{\kappa}$, $\boldsymbol{\beta}$) with the surgery link k + **o**. Since $P\left(F_0^1\right)$ is equivalent to F in $\mathbf{R}_+^4$ and ucl$\left(F_0^1\right)'$ is a ribbon surface with $\partial ucl\left(F_0^1\right)' = \partial F = k$, there is a boundary-relative renewal embedding from ucl $\left(F_0^1\right)'$ to F. This completes the proof of the classical ribbon theorem.

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