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Research Article

Theory of Nonlinear Waves in Plasma with a Constant Magnetic Field

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Abstract

In the paper, for the first time, the analytic solution of the collisionless time-dependent Vlasov-Boltzmann kinetic equation in the presence of a constant magnetic field. By means of this new solution, the general electrodynamic equation, describing the behavior of waves, also in the nonlinear stage, is considered. The analysis is restricted to the consideration of the first nonlinear approximation, keeping the second power of the electric strength. As it is known, the plasma with a magnetic field is characterized by the abundance of different wave branches, describing which demands rather long and tedious calculations. We restrict ourselves to the consideration of perturbations propagating along the constant magnetic field. Obviously, any modification of the distribution function can lead to the essential change of wave-like behavior of plasma and the whole plasma electrodynamics. It is shown that in the nonlinear stage, the longitudinal and transversal waves cannot propagate independently – they are coupled with each other.

More Information

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1. Introduction

A large number of papers and textbooks are devoted to the theory of the waves in the magnetized plasma [1-3]. Possible branches of oscillations in the magnetized plasma allows to consider different approaches and interpretations for waves [4,5]. In the predominant part of published papers, the solution of the linearized Vlasov-Boltzmann kinetic equation is used.

After finding the zero-order solution of the main equation, the authors construct approximations for any higher order. In most parts of books on the kinetic theory of gases, also in original papers, one cannot observe attempts to immediately determine the solution of the Vlasov-Boltzmann kinetic equation [6,7]. In papers [8-10] a general solution of the Vlasov-Boltzmann kinetic equation is presented, but this solution is connected with the infinite chain of approximations, which should be cut off.

In the present paper, for the first time, the exact analytic solution of the time-dependent Vlasov-Boltzmann kinetic equation for plasma with a constant magnetic field. It is known that all wave-like behaviors (also the electromagnetic characteristics) of plasma are determined by the distribution function (DF) of charged particles. In the presence of the magnetic field, a high number of possibilities exists for the investigation of different branches of waves.

Using the time-dependent DF in the present paper, only waves propagating along the constant magnetic field as a first step are considered. The description of waves with other directions we plan for the future, as they are connected with tedious and long calculations. The nonlinearity of waves, expressed in the form of the second power of the electric field, is taken into account. It should be emphasized that the nonlinearity of waves in a magnetoactive plasma is rarely discussed in the literature [1,4,5].

2. Exact solution of the Vlasov-Boltzmann kinetic equation

We start from the collisionless Vlasov-Boltzmann equation for particles of α-type,

$$\frac{\partial f_\alpha}{\partial t}+\vec{v}\frac{\partial f_\alpha}{\partial \vec{r}}+\frac{Z_\alpha e}{m_\alpha}\left\{\vec{E}(\vec{r},t)+\frac{1}{c}\left[\vec{v}\times\left(\vec{B}(\vec{r},t)+\vec{B}_0\right)\right]\right\}\frac{\partial f_\alpha}{\partial \vec{v}}=0, \tag{1}$$

where $\vec{B}_0$ is the constant magnetic field, Z_α is the number of elementary charges e possessed by the particle (for instance, for electrons Z_e =-1). Using notions of vector $\vec{A}(\vec{r},t), \vec{A}_0(\vec{r})$ and scalar $\varphi(\vec{r},t)$ potentials for the sum of fields in the Lorentz force, we obtain

$$\vec{E}(\vec{r},t)+\frac{1}{c}\left[\vec{v}\times\left(\vec{B}(\vec{r},t)+\vec{B}_0\right)\right]=-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}-\vec{\nabla}\varphi+\frac{1}{c}\left[\vec{v}\times\left[\vec{\nabla}\times\left(\vec{A}+\vec{A}_0\right)\right]\right] \tag{2}$$

Using the relation, one can give the relation (2) the form [11]:

$$\begin{aligned}\vec{E}(\vec{r},t)+\frac{1}{c}\left[\vec{v}\times\left(\vec{B}(\vec{r},t)+\vec{B}_0\right)\right]=-\frac{d}{dt}\left\{\int^{t}dt'\left\{\frac{1}{c}\frac{\partial}{\partial t'}\vec{A}(\vec{r},t')+\vec{\nabla}\phi(\vec{r},t')\right\}+\frac{1}{c}\vec{A}_0(\vec{r})\right\}\\+v_i\cdot\vec{\nabla}\cdot\left\{\int^{t}dt'\left\{\frac{1}{c}\frac{\partial}{\partial t'}A_i(\vec{r},t')+\nabla_i\varphi(\vec{r},t')\right\}+\frac{1}{c}A_{0_i}(\vec{r})\right\}.\end{aligned} \tag{3}$$

Lower limits of integrals will be determined by assumption that the electric field is switched on at the initial moment t_0:

$$\vec{\varepsilon}_\alpha(\vec{r},t)=\frac{Z_\alpha e}{m_\alpha}\left\{\int_{t_0}^{t}dt'\cdot\vec{E}(\vec{r},t')-\frac{1}{c}\vec{A}_0(\vec{r})\right\}, \tag{4}$$

where $\vec{E}(\vec{r},t)=-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r},t)-\vec{\nabla}\varphi(\vec{r},t)$. The kinetic equation (1) acquires the form

$$\frac{\partial f_\alpha}{\partial t}+\vec{v}\frac{\partial f_\alpha}{\partial\vec{r}}+\frac{d}{dt}\vec{\varepsilon}_\alpha(\vec{r},t)\cdot\frac{\partial f_\alpha}{\partial\vec{v}}-\left(v_i\cdot\vec{\nabla}\cdot\varepsilon_{\alpha i}(\vec{r},t)\right)\cdot\frac{\partial f_\alpha}{\partial\vec{v}}=0\,. \tag{5}$$

Afterwards, it is convenient to introduce a new velocity variable, $\vec{w}=\vec{v}-\vec{\varepsilon}_\alpha(\vec{r},t)$, ($\vec{v}=\vec{w}+\vec{\varepsilon}_\alpha(\vec{r},t)$), that implies the change of derivatives in the equation (5),

$$\frac{\partial f_\alpha(t,\vec{r},\vec{v})}{\partial t}\to\frac{\partial f_\alpha(t,\vec{r},\vec{w})}{\partial t}-\frac{\partial}{\partial t}\vec{\varepsilon}_\alpha(\vec{r},t)\cdot\frac{\partial f_\alpha(t,\vec{r},\vec{w})}{\partial\vec{w}}, \tag{6}$$

$$\frac{\partial f_\alpha(t,\vec{r},\vec{v})}{\partial r_i}\to\frac{\partial f_\alpha(t,\vec{r},\vec{w})}{\partial r_i}-\frac{\partial}{\partial r_i}\vec{\varepsilon}_\alpha(\vec{r},t)\cdot\frac{\partial f_\alpha(t,\vec{r},\vec{w})}{\partial\vec{w}}, \tag{7}$$

which results in the compact form of the kinetic equation (5)

$$\frac{\partial f_\alpha}{\partial t}+[H\cdot f_\alpha]=0, \tag{8}$$

where $[H\cdot f_\alpha]=\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{w}}\cdot\frac{\partial f_\alpha}{\partial\vec{r}}-\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{r}}\cdot\frac{\partial f_\alpha}{\partial\vec{w}}$ (9)

Is the Poisson's brackets defined by means of the kinetic energy,

$$H(t,\vec{r},\vec{w})=\frac{1}{2}\{\vec{w}+\vec{\varepsilon}_\alpha(t,\vec{r})\}^2\equiv H(t)\cdot \tag{10}$$

In the following, for shortening, we sometimes use the notation $H(t)$. The characteristic equations for Eq. (8) read

$$\frac{d\vec{w}}{dt}=-\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{r}}, \tag{11}$$

$$\frac{d\vec{r}}{dt}=\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{w}}. \tag{12}$$

Further calculations are carried out for the characteristic equation only for $\vec{w}$. Results for the characteristic equation for the coordinate $\vec{r}$ (see equation (12)) can be obtained by a simple change of notations. From equation (11) we obtain

$$\frac{d\vec{w}}{dt}=-\frac{\partial}{\partial t}\int^{t}dt'\frac{\partial H(t',\vec{r},\vec{w})}{\partial\vec{r}}=-\frac{d}{dt}\int^{t}dt'\frac{\partial H(t',\vec{r},\vec{w})}{\partial\vec{r}}+\left(\frac{d\vec{r}}{dt}\cdot\frac{\partial}{\partial\vec{r}}\right)\int^{t}dt'\frac{\partial H(t',\vec{r},\vec{w})}{\partial\vec{r}}+\left(\frac{d\vec{w}}{dt}\cdot\frac{\partial}{\partial\vec{w}}\right)\int^{t}dt'\frac{\partial H(t',\vec{r},\vec{w})}{\partial\vec{r}}$$

$$-\frac{d}{dt}\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}+\left(\frac{dH(t)}{d\vec{w}}\cdot\frac{\partial}{\partial\vec{r}}\right)\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}-\left(\frac{dH(t)}{d\vec{r}}\cdot\frac{\partial}{\partial\vec{w}}\right)\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}. \tag{13}$$

Introducing the operator $\{\hat{t}\}=\frac{d\vec{r}}{dt}\cdot\frac{\partial}{\partial\vec{r}}+\frac{d\vec{w}}{dt}\cdot\frac{\partial}{\partial\vec{r}}=\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{w}}\cdot\frac{\partial}{\partial\vec{r}}-\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{w}}\cdot\frac{\partial}{\partial\vec{r}}$ (14)

and continue the transformation, made for obtaining Eq. (13), we find

$$\frac{d\vec{w}}{dt}=-\frac{d}{dt}\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}+\{\hat{t}\}\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}=-\frac{d}{dt}\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}+\frac{d}{dt}\int^{t} dt'\cdot\{\hat{t}'\}\int^{t'} dt''\frac{\partial H(t'')}{\partial\vec{r}}-\{\hat{t}\}\int^{t} dt'\cdot\{\hat{t}'\}\int^{t'} dt''\frac{\partial H(t'')}{\partial\vec{r}} \tag{15}$$

Last term in the right-hand (rh) side of Eq.(15), by analogy to preceding terms, can be also transformed to the form with the full time derivative. The similar procedure can be realized for all subsequent terms. In consequence we obtain the expression for the first constant integral for Eq.(1):

$$\vec{U}=\vec{w}+\int^{t} dt'\frac{\partial H(t')}{\partial\vec{r}}-\int^{t} dt'\cdot\{\hat{t}'\}\int^{t'} dt''\frac{\partial H(t'')}{\partial\vec{r}}+\int^{t} dt'\cdot\{\hat{t}'\}\int^{t'} dt''\{\hat{t}''\}\int^{t''} dt'''\frac{\partial H(t''')}{\partial\vec{r}}-\int^{t} dt'\cdot\{\hat{t}'\}\int^{t'} dt''\{\hat{t}''\}\int^{t''} dt'''\{\hat{t}'''\}\int^{t'''} dt^{IV}\frac{\partial H(t^{IV})}{\partial\vec{r}}+ \tag{16}$$

After using the transformation of integrals,

$$\int^{t} dt'\int^{t'} dt''\ldots=\int^{t} dt''\int_{t''}^{t} dt'\ldots.' \tag{17}$$

the expression (16) acquires the form

$$\vec{U}=\vec{w}+\int^{t} dt'\left\{1-\int_{t'}^{t} dt''\cdot\{t''\}+\frac{1}{2}\left\{\int_{t'}^{t} dt''\cdot\{t''\}\right\}^{2}-\frac{1}{6}\left\{\int_{t'}^{t} dt''\cdot\{t''\}\right\}^{3}+\ldots\right\}\cdot\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{r}} \tag{18}$$

For obtaining (18) and (19) see **Appendix A.** Expression for the second constant of integration $\vec{R}$ can be obtained by means of a change of variables in equations (12), (13), and (18):

$$\vec{R}=\vec{r}-\int^{t} dt'\left\{1-\int_{t'}^{t} dt''\cdot\{t''\}+\frac{1}{2}\left\{\int_{t'}^{t} dt''\cdot\{t''\}\right\}^{2}-\frac{1}{6}\left\{\int_{t'}^{t} dt''\cdot\{t''\}\right\}^{3}+\ldots\right\}\cdot\frac{\partial H(t,\vec{r},\vec{w})}{\partial\vec{w}}. \tag{19}$$

Expressions (18) and (19) represent expansions of the following relations

$$\vec{U}=\vec{w}+\int^{t} dt'\frac{\partial}{\partial\vec{r}}H\left\{t',\vec{r}-\int_{t'}^{t} dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}},\vec{w}+\int_{t'}^{t} dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{r}}\right\}, \tag{20}$$

$$\vec{R}=\vec{r}-\int^{t} dt'\frac{\partial}{\partial\vec{w}}H\left\{t',\vec{r}-\int_{t'}^{t} dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}},\vec{w}+\int_{t'}^{t} dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{r}}\right\}. \tag{21}$$

By direct substitution one can easily be convinced that the expression,

$$f_\alpha=f_\alpha\left\{\vec{R}(t,\vec{r},\vec{w}),\vec{U}(t,\vec{r},\vec{w})\right\}, \tag{22}$$

together with Eqs. (20) and (21) is the solutions of the Vlasov-Boltzmann equation (1). Indeed from Eqs.(8) and (9) we can construct the identical equation:

$$\frac{\partial f_\alpha}{\partial\vec{R}}\left\{\frac{\partial\vec{R}}{\partial t}+\frac{\partial H}{\partial\vec{w}}\frac{\partial\vec{R}}{\partial\vec{r}}-\frac{\partial H}{\partial\vec{r}}\frac{\partial\vec{R}}{\partial\vec{w}}\right\}+\frac{\partial f_\alpha}{\partial\vec{U}}\left\{\frac{\partial\vec{U}}{\partial t}+\frac{\partial H}{\partial\vec{w}}\frac{\partial\vec{U}}{\partial\vec{r}}-\frac{\partial H}{\partial\vec{r}}\frac{\partial\vec{U}}{\partial\vec{w}}\right\}=0. \tag{23}$$

Taking derivatives with time from equations (20) and (21) we find,

$$\frac{\partial\vec{R}}{\partial t}=-\{\hat{t}\}\cdot\vec{R}\ \text{ and }\ \frac{\partial\vec{U}}{\partial t}=-\{\hat{t}\}\cdot\vec{U} \tag{24}$$

Hence the distribution function of α-particles then should be presented in the form (22). For further analysis, we need to determine inverse solutions for $\vec{r}$ and $\vec{w}$ of Eqs.(20) and (21). From these equations, it follows that [11]:

$$\vec{r}-\int_{t}^{t} dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}}=\vec{\rho}(t)=\vec{R}+\int^{t} dt'\frac{\partial}{\partial\vec{w}}H\{\vec{\rho}(t'),\vec{\omega}(t')\},$$

$$\vec{r}-\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}}=\vec{\rho}(t')=\vec{R}+\int^{t'}dt''\frac{\partial}{\partial\vec{w}}H\{\vec{\rho}(t''),\vec{\omega}(t'')\},$$

$$\vec{r}-\int_{t''}^{t'}dt'''\frac{\partial H(t''',\vec{r},\vec{w})}{\partial\vec{w}}=\vec{\rho}(t'')=\vec{R}+\int^{t''}dt'''\frac{\partial}{\partial\vec{w}}H\{\vec{\rho}(t'''),\vec{\omega}(t''')\}, \tag{25}$$

$$\vec{w}+\int_{t}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{r}}=\vec{\omega}(t)=\vec{U}-\int^{t}dt'\frac{\partial}{\partial\vec{r}}H\{\vec{\rho}(t'),\vec{\omega}(t')\}, \tag{26}$$

The chain of expressions (25, 25’,.....) and (26, 26’,......) can be continued which allows to present inverse solutions in the following form:

$$\vec{r}=\vec{R}+\int^{t}dt'\frac{\partial}{\partial\vec{U}}H\left\{t',\vec{R}+\int^{t'}dt''\frac{\partial}{\partial\vec{U}}H\{t'',\vec{\rho}(t''),\ \vec{\omega}(t'')\},\vec{U}-\int^{t'}dt''\frac{\partial}{\partial\vec{R}}H\{t'',\vec{\rho}(t''),\vec{\omega}(t'')\}\right\}, \tag{27}$$

$$\vec{w}=\vec{U}-\int^{t}dt'\frac{\partial}{\partial\vec{R}}H\left\{t',\vec{R}+\int^{t'}dt''\frac{\partial}{\partial\vec{U}}H\{t'',\vec{\rho}(t''),\vec{\omega}(t'')\},\vec{U}-\int^{t'}dt''\frac{\partial}{\partial\vec{R}}H\{t'',\vec{\rho}(t''),\vec{\omega}(t'')\}\right\}. \tag{28}$$

By successively substituting $\vec{\rho}(t''),\vec{\rho}(t'''),....,$ and $\vec{\omega}(t''),\vec{\omega}(t'''),...$ into solutions (27) and (28), these solutions take the form of an infinite series of the constants of integration $\vec{R}$ and $\vec{U}$ in the arguments of functions $H\{t'',\vec{\rho}(t''),\vec{\omega}(t'')\},H\{t''',\vec{\rho}(t'''),\vec{\omega}(t''')\},....$. Solutions (27) and (28) we’ll use below.

We define the distribution function (DF) assuming that at the initial moment, t_0, the DF of α–particles depends only on the velocity

$$f_\alpha\{\vec{R}(t_0,\vec{r},\vec{v}),\vec{U}(t_0,\vec{r},\vec{v})\}=f_{0\alpha}(v)\big|_{t_0} \tag{29}$$

From relations at the initial time t_0 it follows:,

$$\vec{R}(t_0,\vec{r},\vec{w})=R \text{ and } \vec{U}(t_0,\vec{r},\vec{w})=U \tag{30}$$

where $\vec{w}=\vec{v}-\vec{\varepsilon}(\vec{r},t_0)$, and $\vec{\varepsilon}(\vec{r},t)$ is determined by Eq. (4). Then by means of Eq. (28) we can find expression for the velocity $\vec{v}$ at the moment t_0:

$$\vec{v}=-\frac{Z_\alpha e}{m_\alpha}\frac{1}{c}\vec{A}_0++\vec{U}-\int^{t_0}dt'\frac{\partial}{\partial\vec{R}}H\left\{\vec{R}+\int^{t'}dt''\frac{\partial}{\partial\vec{U}}H\{\vec{R}+....,\vec{U}-....\},\vec{U}-\int^{t'}dt''\frac{\partial}{\partial\vec{R}}H\{\vec{R}+....,\vec{U}-....\}\right\} \tag{31}$$

(31) Substituting the relation (20) with $\vec{w}=\vec{v}-\vec{\varepsilon}_\alpha(\vec{r},t)$ into the right-hand side of Eq.(31) we obtain an argument for the distribution function,

$$\vec{v}-\frac{Z_\alpha e}{m_\alpha}\int_{t_0}^{t}dt'\cdot\vec{E}(\vec{r},t)++\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec{r}}H\left\{t',\vec{r}-\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}},\vec{w}+\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{r}}\right\} \tag{32}$$

Satisfying the condition (29). Hence for the DF of α-particles we obtain

$$f_\alpha=f_{0\alpha}\left\{\vec{v}-\frac{Z_\alpha e}{m_\alpha}\int_{t_0}^{t}dt'\cdot\vec{E}(\vec{r},t)++\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec{r}}H\left\{t',\vec{r}-\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}},\vec{w}+\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{r}}\right\}\right\} \tag{33}$$

As $f_{0\alpha}(\vec{v})$ we can choose the Maxwell distribution function with normalizing coefficient, $\left(1/\sqrt{2\pi}\right)^{3/2}\cdot\exp\left(-v^2/2\right)$:

$$f_\alpha=n_{0\alpha}\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2}\exp\left\{-\frac{m_\alpha}{2T_\alpha}\left[\vec{v}-\frac{Z_\alpha e}{m_\alpha}\int_{t_0}^{t}dt'\cdot\vec{E}(\vec{r},t)++\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec{r}}H\left\{t',\vec{r}-\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{w}},\vec{w}+\int_{t'}^{t}dt''\frac{\partial H(t'',\vec{r},\vec{w})}{\partial\vec{r}}\right\}\right]^2\right\} \tag{34}$$

In the following we’ll restrict ourselves taking into account the electric strength up to second power inclusive. Using expansions (18), and relations (10), (14) we find

$$f_\alpha=n_{0\alpha}\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2}\exp\left\{-\frac{m_\alpha}{2T_\alpha}\left[\vec{w}-\frac{Z_\alpha e}{m_\alpha}\vec{A}_0(\vec{r})+\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec{r}}\frac{1}{2}\{\vec{w}+\vec{\varepsilon}_\alpha(\vec{r},t')\}^2-\int_{t_0}^{t}dt'\cdot\int_{t'}^{t}dt''\left\{\frac{\partial}{\partial\vec{w}}\frac{1}{2}\{\vec{w}+\vec{\varepsilon}_\alpha(\vec{r},t'')\}^2\cdot\frac{\partial}{\partial\vec{r}}-\frac{\partial}{\partial\vec{r}}\frac{1}{2}\{\vec{w}+\vec{\varepsilon}_\alpha(\vec{r},t'')\}^2\cdot\frac{\partial}{\partial\vec{w}}\right\}\frac{\partial}{\partial\vec{r}}\frac{1}{2}\{\vec{w}+\vec{\varepsilon}_\alpha(\vec{r},t')\}^2\right]^2\right\} \tag{35}$$

It is convenient to introduce farther new variables instead of (4):

$$\vec{\varepsilon}_\alpha(\vec{r},t)=\vec{\sigma}_\alpha(\vec{r},t)-\vec{a}_\alpha(\vec{r}), \tag{36}$$

$$\vec{\sigma}_\alpha(\vec{r},t)=\frac{Z_\alpha e}{m_\alpha}\int_{t_0}^{t}dt'\cdot\vec{E}(\vec{r},t'),\ \vec{a}_\alpha(\vec{r})=\frac{Z_\alpha e}{m_\alpha c}\vec{A}_0(\vec{r}), \tag{37}$$

Substituting relations $\vec{w}=\vec{v}-\vec{\varepsilon}_\alpha(\vec{r},t)$ and (36) into Eq.(35) (choosing $\vec{\varepsilon}_\alpha(\vec{r},t)$ for different time-moments $t',t'',...$) we obtain

$$f_\alpha(t,\vec r,\vec v)=n_{0\alpha}\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2}\exp\left\{-\frac{m_\alpha}{2T_\alpha}\left[\vec v-\vec\sigma_\alpha(\vec r,t)+\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\frac{1}{2}\left\{\vec v-\vec\sigma_\alpha(\vec r,t)+\vec\sigma_\alpha(\vec r,t')\right\}^2-\right.\right.$$
$$\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\frac{1}{2}\left\{\vec v-\vec\sigma_\alpha(\vec r,t)+\vec\sigma_\alpha(\vec r,t')\right\}^2-\int_{t_0}^{t}dt'\int_{t'}^{t}dt''\left\{\frac{\partial}{\partial\vec v}\frac{1}{2}\left(\vec v-\vec\sigma_\alpha(\vec r,t)+\vec\sigma_\alpha(\vec r,t'')\right)^2\frac{\partial}{\partial\vec r}-\right.$$
$$\left.\left.\left.\frac{\partial}{\partial\vec r}\frac{1}{2}\left(\vec v-\vec\sigma_\alpha(\vec r,t)+\vec\sigma_\alpha(\vec r,t'')\right)^2\frac{\partial}{\partial\vec v}\right\}\cdot\frac{\partial}{\partial\vec r}\frac{1}{2}\left\{\vec v-\vec\sigma_\alpha(\vec r,t)+\vec\sigma_\alpha(\vec r,t')\right\}^2\right]^2\right\} \tag{38}$$

After squaring all terms indicated and expansion of the exponential function up to the second order of the electric strength, we obtain

$$f_\alpha(t,\vec r,\vec u+\vec\sigma_\alpha(\vec r,t))=n_{0\alpha}\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2}\exp\left(-\frac{m_\alpha}{2T_\alpha}u^2\right)\left\{1-\frac{m_\alpha}{2T_\alpha}u_i\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\vec\sigma_{\alpha i}(\vec r,t').u_k\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\vec\sigma_{\alpha k}(\vec r,t')-\right.$$
$$\frac{m_\alpha}{T_\alpha}\vec u\cdot u_i\left(\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\sigma_{\alpha i}(\vec r,t')-\int_{t_0}^{t}dt'\int_{t'}^{t}dt''\frac{\partial}{\partial r_i}.\sigma_{\alpha k}(\vec r,t'')\frac{\partial}{\partial\vec r}\sigma_{\alpha k}(\vec r,t')-\int_{t_0}^{t}dt'\int_{t'}^{t}dt''\sigma_{\alpha k}(\vec r,t'')\frac{\partial}{\partial r_k}\cdot\frac{\partial}{\partial\vec r}\sigma_{\alpha i}(\vec r,t')+\int_{t_0}^{t}dt'\int_{t'}^{t}dt''\frac{\partial}{\partial r_k}\sigma_{\alpha i}(\vec r,t'')\cdot\frac{\partial}{\partial\vec r}\sigma_{\alpha k}(\vec r,t')\right)+$$
$$+\frac{m_\alpha}{T_\alpha}\vec u\cdot u_i\cdot u_k\frac{\partial}{\partial r_i}\frac{\partial}{\partial\vec r}\int_{t_0}^{t}dt'\int_{t'}^{t}dt''\sigma_{\alpha k}(\vec r,t')-\frac{m_\alpha}{T_\alpha}\vec u\int_{t_0}^{t}dt'\ \sigma_{\alpha k}(\vec r,t')\cdot\frac{\partial}{\partial\vec r}\sigma_{\alpha k}(\vec r,t')$$
$$\left.+\frac{1}{2}\left(\frac{m_\alpha}{T_\alpha}\right)^2\vec u\cdot u_i\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\vec\sigma_{\alpha i}(\vec r,t')\cdot\vec u\cdot u_k\int_{t_0}^{t}dt'\frac{\partial}{\partial\vec r}\sigma_{\alpha k}(\vec r,t')\right\}. \tag{39}$$

where the shifting velocity is introduced $\vec u=\vec v-\vec\sigma_\alpha(\vec r,t)$.

3. General expression for the current and its time-derivative

Analysis of waves we start from the main electrodynamic equation

$$rot\cdot rot\cdot\vec E+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec E+\frac{4\pi}{c^2}\sum_\alpha Z_\alpha e\frac{\partial}{\partial t}\vec J_\alpha(\vec r,t)=0, \tag{40}$$

where the current $\vec J_\alpha(\vec r,t)$ should be defined by means of Eq. (35),

$$\vec J_\alpha(\vec r,t)=\int_{-\infty}^{\infty}d\vec v\cdot\vec v\cdot f_{0\alpha}(t,\vec r,\vec v)=\int_{-\infty}^{\infty}d\vec u\cdot\left\{\vec u+\vec\sigma_\alpha(\vec r,t)\right\}\cdot f_{0\alpha}(t,\vec r,\vec u+\vec\sigma_\alpha(\vec r,t)) \tag{41}$$

According to equations (38) and (39) one can be convinced that the constant magnetic field falls out of calculations. It means that the replacement of variables (realization of which is strictly justified for expressions under the integral) can eliminate the constant magnetic field from visual consideration, leaving all definitions obtained before that, valid. This result is apparently analogous to the situation, when in a system described by a "pure" Maxwell distribution function (DF), the force associated with the magnetic field disappears. In our case, the generalised expression of the Maxwell DF does not contain the constant magnetic field. Hence further investigation can be continued by means of the DF (38) and (39), since they are valid in the presence of a constant magnetic field. Obviously, one should only transform the argument of the DF as it is done in Eqs. (38) and (39). This leads to the necessity to change the interpretation of the initial equation. Equation (1) we rewrite in the form

$$\frac{\partial f_\alpha}{\partial t}+\vec v\frac{\partial f_\alpha}{\partial\vec r}+\frac{Z_\alpha e}{m_\alpha}\left\{\vec E(\vec r,t)+\frac{1}{c}\left[\vec v\times\left(\vec B(\vec r,t)\right)\right]\right\}\frac{\partial f_\alpha}{\partial\vec v}+\left[\vec v\times\vec\Omega_\alpha\right]\frac{\partial f_\alpha}{\partial\vec v}=0, \tag{42}$$

where $\vec\Omega_\alpha=\frac{Z_\alpha e}{m_\alpha c}\vec B_0$. The expression (2), represented without the constant magnetic field $\vec B_0$, after repeating calculations analogous to those used for obtaining Eqs. (3)-(5), acquires the form:

$$\frac{Z_\alpha e}{m_\alpha}\left\{\vec E(\vec r,t)+\frac{1}{c}\left[\vec v\times\vec B(\vec r,t)\right]\right\}=\frac{d}{dt}\vec\sigma_\alpha(\vec r,t)+\vec v\cdot\vec\nabla\cdot\vec\sigma_\alpha(\vec r,t), \tag{43}$$

where $\vec\sigma_\alpha(\vec r,t)$ is defined by Eq.(37). Then, for the kinetic equation (42), we obtain

$$\frac{\partial f_\alpha}{\partial t}+\vec v\frac{\partial f_\alpha}{\partial\vec r}+\frac{\partial\vec\sigma_\alpha(\vec r,t)}{\partial t}\frac{\partial f_\alpha}{\partial\vec v}-\left[\vec v\left[\vec\nabla\times\vec\sigma_\alpha(\vec r,t)\right]\right]\frac{\partial f_\alpha}{\partial\vec v}+\left[\vec v\times\vec\Omega_\alpha\right]\frac{\partial f_\alpha}{\partial\vec v}=0. \tag{44}$$

Using the relation (41) for the derivative of the current from Eq. (44), we find

$$\frac{\partial}{\partial t}\vec{J}_\alpha = -\frac{\partial}{\partial r_k}\int d\vec{v}\cdot\vec{v}\cdot v_k f_{0\alpha} + \frac{\partial\vec{\sigma}_\alpha}{\partial t}\int d\vec{v}\cdot f_{0\alpha} - \left[\vec{J}_\alpha\left[\vec{\nabla}\times\vec{\sigma}_\alpha\right]\right] + \left[\vec{J}_\alpha\times\vec{\Omega}_\alpha\right]. \tag{45}$$

Further detailed calculations are given in **Appendix B.** Substituting the relation (**A5**) from **Appendix B** into (45), we obtain the main equation for the analysis of oscillatory behaviour of plasma:

$$rot\cdot rot\cdot\vec{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} + \frac{1}{c^2}\sum_\alpha \omega_{p\alpha}^2 + \left\{\vec{E}(\vec{r},t) + V_{T_\alpha}^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\left(\Delta\vec{E}(\vec{r},t'') + 2\vec{\nabla}div\vec{E}(\vec{r},t'')\right)\right.$$

$$+\left[\left\{\int_{t_0}^{t} dt'\vec{E}(\vec{r},t') + V_{T_\alpha}^2\int_{t_0}^{t} dt'\int_{t'}^{t} dt''\int_{t_0}^{t''} dt'''\left(\Delta\vec{E}(\vec{r},t''') + 2\vec{\nabla}div\vec{E}(\vec{r},t''')\right)\right\}\times\vec{\Omega}_\alpha\right]\Bigg\}$$

$$-\frac{1}{c^2}\sum_\alpha \omega_{p\alpha}^2\frac{Z_\alpha e}{m_\alpha}\left\{3\frac{\partial}{\partial r_k}\cdot\int_{t_0}^{t} dt' E_{\alpha k}(\vec{r},t')\int_{t_0}^{t} dt'\vec{E}_\alpha(\vec{r},t') + \vec{E}(\vec{r},t)\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' div\vec{E}(\vec{r},t') + \left[\int_{t_0}^{t} dt'\vec{E}(\vec{r},t')\left[\vec{\nabla}\times\int_{t_0}^{t} dt'\vec{E}(\vec{r},t')\right]\right] + \right.$$

$$\left.\left[\left\{\int_{t_0}^{t} dt'\cdot\int_{t_0}^{t'} dt'' E_k(\vec{r},t'')\frac{\partial}{\partial\vec{r}}\int_{t_0}^{t'} dt'' E_k(\vec{r},t'') + \int_{t_0}^{t} dt' E(\vec{r},t')\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' div\vec{E}(\vec{r},t'')\right\}\times\vec{\Omega}_\alpha\right]\right\} = 0, \tag{46}$$

Where is the Langmuir frequency of α-particles?

4. Waves propagating along the constant magnetic field

As it is mentioned above, plasma in a magnetic field is characterized by a great number of different types of waves [1,4,5]. Here, we consider longitudinal and transversal waves propagating along the magnetic field (z – direction).

(a) Longitudinal waves.

Obviously, for longitudinal waves we can assume that $\sum_\alpha \omega_{p\alpha}^2 \approx \omega_{pe}^2$, where is the electron Langmuir frequency. After taking the second derivative with time from Eq. (46) for $E_z(z,t)$ the component of the electric field, we find

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2}{\partial t^2}E_z + \omega_p^2 E_z\right\} + \omega_p^2\, 3V_{T_e}^2\frac{\partial^2}{\partial z^2}E_z + \omega_p^2\frac{e}{m_e}\frac{\partial^2}{\partial t^2}\left\{3\frac{\partial}{\partial z}\cdot\int_{t_0}^{t} dt' E_z(z,t')\int_{t_0}^{t} dt' E_z(z,t') + \right.$$

$$\left. + E_z(z,t)\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\frac{\partial}{\partial z}E_z(z,t'') + \int_{t_0}^{t} dt'\vec{E}(z,t')\frac{\partial}{\partial z}\int_{t_0}^{t} dt'\vec{E}(z,t') - \int_{t_0}^{t} dt' E_z(z,t')\frac{\partial}{\partial z}\int_{t_0}^{t} dt' E_z(z,t')\right\} = 0 \tag{47}$$

Last-but-one term in Eq.(47) can contain transversal components of the electric field. It means that longitudinal and transversal field components can couple to each other in the nonlinear region. Considering nonlinear terms as small perturbations and split the electric field into two parts $E_z = E_{0z} + \delta E_z$, we can represent Eq. (47) in the form of two equations:

$$_{pe}^{2}\, 3V_{Te}^2\frac{\partial^2}{\partial z^2}\delta E_z = -\omega_{pe}^2\frac{e}{m_e}\frac{\partial^2}{\partial t^2}\left\{3\frac{\partial}{\partial z}\cdot\int_{t_0}^{t} dt'\right. \tag{48}$$

$$\frac{\partial^4}{\partial t^4}\delta E_z + \omega_{pe}^2\frac{\partial^2}{\partial t^2}\delta E_z + \omega_{pe}^2 3V_{Te}^2\frac{\partial^2}{\partial z^2}\delta E_z = -\omega_{pe}^2\frac{e}{m_e}\frac{\partial^2}{\partial t^2}\left\{3\frac{\partial}{\partial z}\cdot\int_{t_0}^{t} dt' E_{0z}(z,t')\int_{t_0}^{t} dt' E_{0z}(z,t') + \right. \tag{49}$$

$$\left. E_{0z}(z,t)\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\frac{\partial}{\partial z}E_{0z}(z,t'') + \int_{t_0}^{t} dt' E_{0x}(z,t')\frac{\partial}{\partial z}\int_{t_0}^{t} dt' E_{0x}(z,t') + \int_{t_0}^{t} dt' E_{0y}(z,t')\frac{\partial}{\partial z}\int_{t_0}^{t} dt' E_{0y}(z,t')\right\}$$

Solution of Eq. (48) reads:

$$E_{0z}(t,z) = A\cdot Cos\,\xi(t,z), \tag{50}$$

where $\xi(t,z) = \sqrt{\frac{2}{3}}\frac{\omega_{pe}}{V_{Te}}\left(\sqrt{3}\cdot V_{T_e}\cdot t - z\right) \equiv \xi(t).$ (51)

The electric field at the initial moment we have chosen as $E_{0z}(t_0,z) = A\cdot Cos\,\xi(t_0,z)$. In variables (51), $\frac{\partial^2}{\partial t^2} = 2\omega_{pe}^2\frac{\partial^2}{\partial\xi^2}$ and Eq.(49) acquires the form:

$$\frac{\partial^2}{\partial \xi^2}\delta E_z(\xi)+\delta E_z(\xi)=\frac{e}{2m_e}\left\{3A^2\sqrt{\frac{2}{3}}\frac{\omega_{pe}}{V_{T_e}}\frac{\partial}{\partial \xi}\left(\int_{\xi_0}^{\xi} d\xi'\cdot Cos\xi'\right)^2-A^2\sqrt{\frac{2}{3}}\frac{\omega_{pe}}{V}Cos\xi\int_{\xi_0}^{\xi} d\xi'\int_{\xi_0}^{\xi'} d\xi''Sin\xi''\right\}. \tag{52}$$

The last two terms in Eq. (49), connected with transversal components, don't give any contributions in subsequent calculations (see **Appendix C**). Variables ξ_0, ξ', ξ'',..... are assumed to be equal

$$\sqrt{\frac{2}{3}}\frac{\omega_{pe}}{V_{Te}}\left(\sqrt{3}\cdot V_{T_e}\cdot t_0-z\right)=\xi_0(t_0),\ \sqrt{\frac{2}{3}}\frac{\omega_{pe}}{V_{Te}}\left(\sqrt{3}V_{Te}\cdot t'-z\right)=\xi'(t'),\ \sqrt{\frac{2}{3}}\frac{\omega_{pe}}{V_{Te}}\left(\sqrt{3}V_{Te}\cdot t''-z\right)=\xi''(t''),$$

similar to Eq. (51). Solution of Eq. (52) represents the nonlinear correction, $\frac{eA}{m_e\omega_p V_{T_e}}<1$, to the linear approximation described by Eq. (50):

$$\delta E_z(\xi)=\frac{1}{2\sqrt{6}}\frac{e}{m_e}\frac{A^2}{V_{T_e}\,\omega_p}\int_{\xi_0}^{\xi} d\xi'\cdot Sin(\xi-\xi_0)\cdot Cos\xi'\cdot\int_{\xi_0}^{\xi'} d\xi''\left\{6\cdot Cos\xi''-\int_{\xi_0}^{\xi''} d\xi'''\cdot Sin\xi'''\right\}. \tag{53}$$

As it is shown in **Appendix D,** the straightforward integrations for Eq. (53) give

$$\begin{aligned}\delta E_z(\xi)=\frac{1}{2\sqrt{6}}\frac{e}{m_e}\frac{A^2}{V_{T_e}\omega_{pe}}\cdot\Big\{&\frac{7}{3}\left(Sin\xi\cdot Cos^3\xi_0+Cos\xi\cdot Sin^3\xi_0-Sin\xi\cdot Cos\xi\right)+\\ &3Sin^2\xi_0\cdot Sin(\xi-\xi_0)+\frac{1}{4}Sin(\xi-\xi_0)\cdot\left\{Sin^2\xi_0+1\right\}-\frac{7}{2}(\xi-\xi_0)\cdot Sin\xi_0\cdot Sin\xi-\\ &\frac{1}{4}(\xi-\xi_0)^2\cdot Cos\xi_0\cdot Sin\xi-\frac{1}{4}(\xi-\xi_0)Cos\xi_0\cdot Cos\xi\Big\}.\end{aligned} \tag{54}$$

(b) Transversal waves.

It looks convenient to describe the transversal waves instead with components of the electric field, but by their combination

$$E_{\pm}=E_x\pm iE_y, \tag{55}$$

Below, we'll consider the electron plasma again. From Eq.(46), $E_{\pm}$ we can construct the equation:

$$\begin{aligned}&\frac{\partial^3}{\partial t^3}E_{\pm}-c^2\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial t}E_{\pm}+\omega_{pe}^2\frac{\partial}{\partial t}E_{\pm}\mp i\omega_{pe}^2\Omega_z E_{\pm}=\omega_{pe}^2\frac{e}{m_e}\Big\{3\frac{\partial E_z}{\partial z}\int_{t_0}^{t}dt'E_{\pm}(z,t')+2E_z\frac{\partial}{\partial z}\int_{t_0}^{t}dt'E_{\pm}(z,t')+3E_{\pm}(z,t)\int_{t_0}^{t}dt'\frac{\partial E_z(z.t')}{\partial z}+\\ &2\int_{t_0}^{t}dt'E_z(z,t')\cdot\frac{\partial}{\partial z}E_{\pm}(z,t)+\frac{\partial E_{\pm}(z,t)}{\partial t}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\frac{\partial E_z(z,t'')}{\partial z}+E_{\pm}(z,t)\int_{t_0}^{t}dt'\frac{\partial E_z(z.t')}{\partial z}\mp\\ &i\left\{E_{\pm}(z,t)\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\frac{\partial E_z(z,t'')}{\partial z}+\int_{t_0}^{t}dt'E_{\pm}(z,t')\cdot\int_{t_0}^{t}dt'\frac{\partial E_z(z.t')}{\partial z}\right\}\times\Omega_z\Big\}.\end{aligned} \tag{56}$$

In obtaining Eq.(56), it was assumed that $c^2>>(\omega_{pe}/\omega)V_{Te}^2$, where is the characteristic frequency of problem considered. The last inequality allows us to exclude the thermal velocity from the subsequent consideration. We'll also assume that the characteristic phase velocity, (ω/k), is much larger than the velocity acquired in the field, $V_E\left(=(e/m_e)\int E(z,t')dt'\right)$, discussed in the present problem, $(\omega/k)>>V_E$. Under this condition, one can split $E_{\pm}$ ($E_{\pm}=E_{0\pm}+\delta E_{\pm}$) and split also the equation (56) into linear and nonlinear parts, as it was made for longitudinal waves:

$$\frac{\partial^3}{\partial t^3}E_{0\pm}-c^2\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial t}E_{0\pm}+\omega_{pe}^2\frac{\partial}{\partial t}E_{0\pm}\mp i\omega_{pe}^2\Omega_z E_{0\pm}=0, \tag{57}$$

$$\frac{\partial^3}{\partial t^3}\delta E_{\pm}-c^2\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial t}\delta E_{\pm}+\omega_{pe}^2\frac{\partial}{\partial t}\delta E_{\pm}\mp i\omega_{pe}^2\Omega_z\delta E_{\pm}=\Phi_{\pm}(z,t), \tag{58}$$

where

$$\Phi_{\pm}(z,t)==-\omega_{pe}^2\frac{e}{m_e}A\frac{\omega_{pe}}{V_{Te}}\sqrt{\frac{2}{3}}\left\{3\cdot Sin(\xi(t))\int_{t_0}^{t}dt'E_{0\pm}(z,t')+\sqrt{6}\frac{V_{Te}}{\omega_{pe}}Cos(\xi(t))\frac{\partial}{\partial z}\int_{t_0}^{t}dt'E_{0\pm}(z,t')+\right.$$

$$=-\omega_p^2\frac{e}{m_e}A\frac{\omega_p}{V_{T}}\sqrt{\frac{2}{3}}\left\{3\cdot Sin(\xi(t))\int_{t_0}^{t}dt'E_{0\pm}(z,t')+\sqrt{6}\frac{V_{T}}{\omega_p}Cos(\xi(t))\frac{\partial}{\partial z}\int_{t_0}^{t}dt'E_{0\pm}(z,t')+\right.$$

$$+3\int_{t_0}^{t} dt' Sin(\xi(t'))\cdot E_{\pm}(z,t)+\sqrt{6}\frac{V_T}{\omega_p}\int_{t_0}^{t} dt'\cdot Cos(\xi(t'))\cdot\frac{\partial}{\partial z}E_{0\pm}(z,t)+$$

$$\frac{\partial E_{0\pm}(z,t)}{\partial t}\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\cdot Sin(\xi(t''))+E_{0\pm}(z,t)\int_{t_0}^{t} dt''\cdot Sin(\xi(t''))\mp$$

$$\mp i\left\{E_{0\pm}(z,t)\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\cdot Sin(\xi(t''))+\int_{t_0}^{t} dt'\cdot E_{o}(\xi(t',z))\cdot\int_{t_0}^{t} dt'\cdot Sin(\xi(t'))\right\}\cdot\Omega_z\bigg\}. \tag{59}$$

On the right-hand side of Eqs. (58) and (59) expression (50) for the main contribution is used. Evidently, solutions of Eqs. (57) and (58) we can consider to be dependent on the variable,

$$\eta(t)=\Omega_z\cdot(t-z/c);\ \eta(t_0)=\Omega_z\cdot(t_0-z/c),\ \eta(t')=\Omega_z\cdot(t'-z/c) \tag{60}$$

Variables $\eta(t_0),\ \eta(t'),\ \eta(t'')$, we need in below. Transforming derivatives to this new variable, Eqs. (57) and (58) acquire the forms

$$\frac{\partial}{\partial\eta}E_{0\pm}\mp E_{0\pm}=0,\ \text{and}\ \frac{\partial}{\partial\eta}\delta E_{\pm}\mp i\delta E_{\pm}=\frac{1}{\omega_p^2\cdot\Omega_z}\Phi(z,t). \tag{61}$$

As solutions of these equations, we find

$$E_{0\pm}(z,t)=C\cdot\exp\{\pm i\Omega_z(t-z/c)\}=C\cdot\exp\{\pm i\cdot\eta\{t\}\}, \tag{62}$$

$$\delta E_{\pm}(z.t)=\exp\{\pm i\eta\}\int_{\eta(t_0)}^{\eta} d\eta'\cdot\exp\{\mp i\eta'\}\cdot\frac{1}{\omega_{pe}^2\cdot\Omega_z}\cdot\Phi_{\pm}(z,t')=\exp\{\pm i\Omega_z\cdot(t-z/c)\}\int_{t_0}^{t} dt'\frac{d\eta'}{dt'}\cdot\exp\{\mp i\Omega_z\cdot(t'-z/c)\}\cdot\frac{1}{\omega_{pe}^2\cdot\Omega_z}\Phi_{\pm}(z,t')=$$

$$\exp\{\pm i\Omega_z(t-z/c)\}\cdot\int_{t_0}^{t} dt'\cdot\exp\{\mp i\Omega_z\cdot(t'-z/c)\}\cdot\frac{1}{\omega_{pez}^2}\Phi_{\pm}(z,t') \tag{63}$$

In descriptive form, for the nonlinear part of the electric field, we have

$$\delta E_{\pm}(z,t)=-\sqrt{\frac{2}{3}}\frac{e}{m_e}A\cdot C\cdot\frac{\omega_{pe}}{V_{Te}}\cdot\exp\{\pm i\eta(t)\}\cdot\int_{t_0}^{t} dt'\left[3\cdot Sin(\xi(t'))\int_{t_0}^{t'} dt''\cdot\exp\{\mp i(\eta(t')-\eta(t''))\}+\right.$$

$$\left.3\int_{t_0}^{t'} dt'' Sin(\xi(t''))+\int_{t_0}^{t'} dt''\cdot Sin(\xi(t''))\mp i\cdot\Omega_z\cdot\int_{t_0}^{t'} dt''\cdot\exp\{\mp i(\eta(t')-\eta(t''))\}\cdot\int_{t_0}^{t'} dt''\cdot Sin(\xi(t''))\right] \tag{64}$$

In obtaining Eq. (64), small terms proportional to are neglected. After rather long calculations for the x-component of the electric field and its correction, following the instructions given in **Appendix E,** we find:

$$E_{0x}(z,t)=C\cdot Cos\left\{\Omega_z\cdot\left(t-\frac{z}{c}\right)\right\}, \tag{65}$$

$$\delta E_x(z,t)=-\frac{2}{\sqrt{3}}\frac{e}{m_e}\frac{A}{\Omega_z\cdot V_{Te}}\cdot C\cdot \tag{66}$$

$$\left(\frac{2}{1}\cdot[Cos\{\eta_+(t)\}\cdot Sin\{\eta_-(t)\}+Sin\{\eta_+(t)+\eta_-(t)\}]\cdot Sin\{\xi_+(t)\}\cdot Sin\{\xi_-(t)\}+\right.$$

$$\frac{3}{\sqrt{2}}\frac{\Omega_z}{\omega_{pe}}\left[Cos\{\xi_+(t)\}\cdot Sin\{\xi_-(t)\}-\frac{1}{\sqrt{2}}\omega_{pe}(t-t_0)Cos\{\xi_+(t)+\xi_-(t)\}\right]\cdot Cos\{\eta_+(t)+\eta_-(t)\}-$$

$$\frac{\sqrt{2}\cdot\omega_{pe}}{\sqrt{2}\cdot\omega_{pe}-\Omega_z}Cos\{\xi_+(t)+\eta_+(t)\}\cdot Sin\{\xi_-(t)-\eta_-(t)\}-$$

$$\left.\frac{\sqrt{2}\cdot\omega_{pe}}{\sqrt{2}\cdot\omega_{pe}+\Omega_z}Cos\{\xi_+(t)-\eta_+(t)\}\cdot Sin\{\xi_-(t)+\eta_-(t)\}\right)$$

and for the **y**-component we find:

$$E_{0y}(z,t) = C \cdot Sin\left\{\Omega_z \cdot \left(t - \frac{z}{c}\right)\right\}, \tag{67}$$

$$\delta E_y(z,t) = -\frac{2}{\sqrt{3}}\frac{e}{m_e}\frac{A}{\Omega_z \cdot V_{Te}} \cdot C \cdot \tag{68}$$

$$\cdot\left(\frac{2}{1}\cdot\left[Sin\{\eta_+(t)\}\cdot Sin\{\eta_-(t)\} - Cos\{\eta_+(t)+\eta_-(t)\}\right]\cdot Sin\{\xi_+(t)\}\cdot Sin\{\xi_-(t)\} -\right.$$

$$-\frac{3}{\sqrt{2}}\frac{\Omega_z}{\omega_{pe}}\left[Cos\{\xi_+(t)\}\cdot Sin\{\xi_-(t)\} - \frac{1}{\sqrt{2}}\omega_{pe}(t-t_0)Cos\{\xi_+(t)+\xi_-(t)\}\right]\cdot Sin\{\eta_+(t)+\eta_-(t)\} -$$

$$+\frac{\sqrt{2}\cdot\omega_{pe}}{\sqrt{2}\cdot\omega_{pe}-\Omega_z}Sin\{\xi_+(t)+\eta_+(t)\}\cdot Sin\{\xi_-(t)-\eta_-(t)\} -$$

$$\left.+\frac{\sqrt{2}\cdot\omega_{pe}}{\sqrt{2}\cdot\omega_{pe}+\Omega_z}Sin\{\xi_+(t)-\eta_+(t)\}\cdot Sin\{\xi_-(t)+\eta_-(t)\}\right).$$

In Eqs. (66) and (68) for brevity, we introduce notations:

$\xi_\pm(t) = \frac{1}{2}\{\xi(t) \pm \xi(t_0)\}$ and $\eta_\pm(t) = \frac{1}{2}\{\eta(t) \pm \eta(t_0)\}$

Definitions of expressions and are given by Eqs. (51) and (69), and the expressions below Eq. (52).

Summary

In the present manuscript, the exact solution of the Vlasov-Boltzmann kinetic equation for plasma with a constant magnetic field is found. The main electrodynamical equation with plasma particles' current is formulated. In the expression of the current nonlinearity, proportional to the second power of the electric strength, is taken into account. Character of longitudinal and transversal waves, also in the nonlinear region, propagating along the constant magnetic field, is described. As expected, the new distribution function of plasma particles, found from the time-dependent kinetic equation,

i) Can lead to a change in the characteristics of waves propagating in the magnetoactive plasma. Instead of the expected equation with second derivatives [1,4,5], we have obtained for longitudinal waves an equation with a time derivative to the fourth power (see Eq. (47)) and for transversal waves an equation with a time derivative to the third power (see Eq. (56)). Reducing these equations to the zero-order approximation (see Eqs. (48) and (57)), as a result we find that longitudinal and transversal waves are characterized by change temporal and spatial parameters (see Eqs. (50) and (62));

ii) Consideration of waves in the nonlinear stage (which is rare in the literature for the magnetized plasma) becomes more visual, which is connected with the clarity of obtaining expressions for particle currents using the found distribution function;

iii) In the nonlinear stage, transversal waves are coupled with the longitudinal waves (see Eqs. (56) and (64)), and hence longitudinal waves affect the character of transversal waves' propagation.

A large number of papers are devoted to waves in a magnetoactive plasma (see [1,4,5] and the literature cited there). But in these papers, attention is mainly paid to linear theories of waves, which is due to difficulties in finding the solution of the kinetic equation, even in the first approximation of the perturbation. In the present manuscript, a complete distribution function (DF) is found that includes nonlinearities (see Eqs. (33) and (34)). This simplifies finding the currents and the fields after expanding the DF. As a first attempt, the wave propagation along the magnetic field is considered, and new results are obtained (see Eqs. (50, 54) and (62, 64)). In the frame of this DF, other cases of waves can also be considered.

Surprisingly, the replacement of integration variables leads to the fall out of terms connected with the constant magnetic field, keeping unchanged the expression for the current.

In the presence of a constant magnetic field, plasma is characterized by different branches of waves. Descriptions of different types of waves and different directions of wave propagation are planned for future investigations.

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